Wiener Algebra Methods for Linear Dispersive Bounds: Some Recent (and Not-Recent) Results

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Wiener L^1 Inversion Lemmas

Theorem (Wiener, 1935)

Suppose $f(x) = \sum_{k=-\infty}^{\infty} a_k e^{ikx}$ *with* $\sum_{k=-\infty}^{\infty} |a_k| < \infty$.

If $f(x) \neq 0$ *at every* $x \in [-\pi, \pi]$, *then*

$$\frac{1}{f(x)} = \sum_{k=-\infty}^{\infty} b_k e^{ikx} \text{ with } \sum_{k=-\infty}^{\infty} |b_k| < \infty.$$

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Identical theorem on $\mathbb{R}$:

Theorem

If $\hat{f} \in L^1(\mathbb{R})$ *and* $1 + f(x) \neq 0$ *at every* $x \in \mathbb{R}$, *then*

$$\mathcal{F}\Big(\frac{1}{1+f(x)} - 1\Big) \in L^1(\mathbb{R}).$$

Dispersive Bounds

For the Schrödinger equation $iu_t + Hu = 0$ with $H = -\Delta + V(x)$ in $\mathbb{R}^n$,
We would like to have the bound

$$\left\| e^{itH} P_{ac}(H) u_0 \right\|_{L^\infty} \lesssim |t|^{-n/2} \|u_0\|_{L^1}. \qquad (*)$$

Question: What short-range condition on V is sufficient?

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In fact, (*) fails if $V(x) = \frac{-c}{|x|^2}$ for any $c > 0$.
[Planchon-Stalker-(Tahvildar-Zadeh), '03]

Using Wiener's Lemma as Written

For the 1D Schrödinger equation, (*) holds with $(1 + |x|)V \in L^1(\mathbb{R})$.

Generic case [G-Schlag, '04],
Zero energy resonance case [Egorova-Kapylova-Marchenko-Teschl, '16]

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Method: Spectral Calculus of H.

$$e^{itH}P_{ac}(H) = \frac{1}{2\pi i}\int_0^\infty e^{it\lambda} \lim_{\epsilon\to 0}\big(\underbrace{(H-(\lambda+i\epsilon))^{-1}}_{R^+(\lambda)} - \underbrace{(H-(\lambda-i\epsilon))^{-1}}_{R^-(\lambda)}\big)\,d\lambda$$

$$= \frac{1}{\pi i}\int_{-\infty}^\infty e^{it\lambda^2}\lambda R^+(\lambda^2)\,d\lambda.$$

Using Wiener's Lemma as Written

Evaluate $\dfrac{1}{\pi i}\displaystyle\int_{-\infty}^{\infty}(e^{it\lambda^2})(\lambda R^+(\lambda^2))\,d\lambda$ using Plancherel identity.

- Fourier transform of $e^{it\lambda^2}$ is bounded by $|t|^{-1/2}$.
- We need Fourier transform of $(\lambda R^+(\lambda^2)(x,y))$ to be in $L^1(\mathbb{R})$ uniformly in x and y.

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Formula for $R^+(\lambda^2)(x,y)$ in terms of scattering data.

$$R^+(\lambda^2)(x,y) = \frac{f^+(\lambda,x)f^-(\lambda,y)}{W(\lambda)}$$

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[GS] used scattering theory results from [Deift-Trubowitz, 1979] about $\hat{f}^+(\,\cdot\,,x)$, $\hat{f}^-(\,\cdot\,,y)$, and $\hat{W}$ being integrable, and $W(\lambda)$ being nonzero.

[EKMT] refined these results in the resonant case.

Using Wiener's Lemma as Written

The discrete Schrödinger equation on $\mathbb{Z}$ has the bound

$$\left\| e^{itH} P_{ac}(H) u_0 \right\|_{\ell^\infty} \lesssim |t|^{-1/3} \|u_0\|_{\ell^1}$$

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Generic case [Cuccagna-Tarulli '09]
Zero energy resonant case [Egorova-Kopylova-Teschl '15]

An Operator-Valued Inversion Theorem

Originated in [Beceanu '11].

Theorem

If $\hat{f}(\rho)$ is an integrable function with values in $B(X)$ which

1. *is norm-continuous with respect to translation,*
2. *can be approximated by compactly supported functions,*

and $I + f(\lambda) \in B(X)$ is invertible for each $\lambda \in \mathbb{R}$,
then the Fourier transform of $[I + f(\lambda)]^{-1} - I$ is also an integrable $B(X)$-valued function.

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More than one definition of "integrable $B(X)$-valued function" is possible. If X is a lattice, and $\hat{f}(\rho)$ are integral operators with kernel $K(\rho, x, y)$, the most versatile definition is to require

$$\int |K(\rho, x, y)|\, d\rho \in B(X).$$

How It Works in 3 Dimensions

For the 3-dimensional Schrödinger equation, one integates $\int_{-\infty}^{\infty} (e^{it\lambda^2}\lambda) R^+(\lambda^2)\, d\lambda$ by parts, then uses the Plancherel identity.

There's a nice operator identity

$$\frac{d}{d\lambda} R^+(\lambda^2) = \underbrace{[I + R_0^+(\lambda)^2 V]^{-1}}_{\substack{\text{need F.T. of this} \\ \text{to be integrable} \\ \text{in } B(L^\infty)}} \underbrace{\frac{d}{d\lambda} R_0^+(\lambda^2)}_{\text{Good!}} \underbrace{[I + VR_0^+(\lambda^2)]^{-1}}_{\substack{\text{need F.T. of this} \\ \text{to be integrable} \\ \text{in } B(L^1)}}$$

How It Works in 3 Dimensions

In three dimensions, $R_0^+(\lambda^2)V(x,y) = \frac{e^{i\lambda|x-y|}V(y)}{4\pi|x-y|}$.

Theorem (Beceanu-G '12)

The conditions of the inversion theorem hold in $B(Ł^\infty)$ *for* $R_0^+(\lambda^2)V$, *and (*) is true provided*

$$\|V\|_{\mathcal{K}} := \sup_y \int_{R^3} \frac{|V(x)|}{|x-y|}\,dx < \infty$$

and V *is in the* $\mathcal{K}$*-closure of* $C_c(\mathbb{R}^3)$.

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and V is in the $\mathcal{K}$-closure of $C_c(\mathbb{R}^3)$.

Actually, condition (1) only holds for $\left(R_0^+(\lambda^2)V\right)^k$ for some large k. But that's good enough.

Higher Order, Higher Dimensions

For the polyharmonic Schrödinger equation with $H = (-\Delta)^m + V(x)$, the main dispersive bounds should be

$$\left\| e^{itH} P_{ac}(H) u_0 \right\|_{L^\infty} \lesssim |t|^{-n/2m} \|u_0\|_{L^1} \qquad (*2)$$

$$\left\| H^{\frac{(m-1)n}{2m}} e^{itH} P_{ac}(H) u_0 \right\|_{L^\infty} \lesssim |t|^{-n/2} \|u_0\|_{L^1} \qquad (*3)$$

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Theorem (Erdogan-G-Green '25)

*Estimates (*2) and (*3) hold in $\mathbb{R}^n$, $2m < n \leq 4m - 1$ provided*

$$\|V\|_{\mathcal{K}^{n-2m}} := \sup_y \int \frac{|V(x)|}{|x-y|^{n-2m}}\, dx < \infty$$

and V is in the $\mathcal{K}^{n-2m}$-closure of $C_c(\mathbb{R}^n)$.

Higher Order, Higher Dimensions

Reasons for the restricted range of n:

- If $n \leq 2m$, then $R_0^+(\lambda^{2m})$ is unbounded as $\lambda \to 0$.
- If $n > 4m - 1$, then $R_0^+(\lambda^{2m})$ is unbounded as $\lambda \to \infty$.

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Additional challenges:

- When $m > 1$, you can't do too much Fourier analysis on $(-\infty, \infty)$ because $R_0^+(\lambda^{2m})$ grows exponentially as $\lambda \to -\infty$.
- The ideal number of times to integrate by parts is $\frac{n-1}{2}$. In even dimensions you have to integrate by parts "one time too many."

An Approach to Low Dimensions

The 1-dimensional Schrödinger equation ($m = 1$, $n = 1$) falls outside the range $2m < n \leq 4m - 1$ specified above.

Never the less, [Hill '20] recovered all the results of [GS] and [EKMT] using an operator-valued inversion argument. The challenges here are:

- $R_0^+(\lambda^2)$ has a rank-1 singularity at $\lambda = 0$.
- After modifying the resolvent to remove the singularity, condition (2) still fails.

Magnetic Schrödinger Operators

The magnetic Schrödinger operator

$$H = -\Delta + \underbrace{i(\vec{A}(x) \cdot \nabla + \nabla \cdot \vec{A}(x)) + V(x)}_{L}$$

has resisted most attempts to prove $L^1 \to L^\infty$ dispersive bounds.

[Beceanu-Kwon, preprint '24] succeeded for a class of potentials in $\mathbb{R}^3$.
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Their strategy can be summarized as:

1. Compute the first $LR_0^+(\lambda^2)$ term with ad-hoc integration by parts.
2. Consider $[I + LR_0^+(\lambda^2)LR_0^+(\lambda^2)]^{-1}$ by pushing both gradients onto the intermediate variable, where it can be integrated away.
3. Deal with all the local integrability problems of $D^2R_0^+(\lambda^2)$.

2-Dimensional Schrödinger Is Stubborn

We can work with scaling-critical potentials in $\mathbb{R}$ and $\mathbb{R}^3$. Why not $\mathbb{R}^2$?

The perennial challenges seem to be:

- The rank-1 singularity of $R_0^+(\lambda^2)$ is of $\log(\lambda)$ type. That's harder to remove gracefully than a simple $\frac{1}{\lambda}$ pole.
- Integrating by parts even once is too much, most of the time.

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Goal: Get a theorem in time for PD27?